

Gapped Quantum Criticality Gains Long Time Quantum Correlations

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Abstract – We show gapped critical environment could remarkably prevent an enhanced decay of decoherence factor and quantum correlations at the critical point, which is nontrivially different from the ones in a gapless critical environment (Quan, et.al Phys. Rev. Lett. **96**, 140604 (2006)). The quantum correlations display very fast decaying to their local minimum at the critical point while maximum decaying occurs away from this point. In particular, our results imply that collapse of decoherence factor is not indicator of a quantum phase transition of environment as opposed to what happens in a gapless criticality. In the weak coupling regime, the relaxation time, at which the quantum correlations touch rapidly local minima, shows a power-law singularity as a function of gap. Furthermore, quantum correlations decay exponentially with second power of relaxation time. Our results are important for a better understanding and characterisation of gap critical environment and its ability as entanglers in open quantum systems.

Quantum correlations (QCs) are of primary importance in quantum information [1, 2] and quantum computation [3–5]. They are related to the basic issue of understanding the nature of non-locality in quantum mechanics [6, 7]. Quantum systems used in quantum information processing inevitably interact with the surrounding environment. These correlated surrounding systems induce quantum decoherency which plays a key role in the understanding of the quantum to the classical transition [8, 9]. As a result, in the last decade a lot of efforts have been devoted to investigate QCs dynamics and decoherence factors of central systems in various environments [10, 11]. The decoherence of a system coupled to a spin environment with quantum phase transition (QPT) has been investigated intensively in various studies [11–20]. Quan et al. [21] considered induction of the Ising-type correlated environment on the Loschmidt echo (LE), and found that the decaying behavior of LE is best enhanced by gapless QPT of the surrounding system. Rossini et al. [22] depicted that in

the short time region the LE decays as a Gaussian. However for long time limits they found that it approaches an asymptotic value, which strongly depends on the strength of the transverse magnetic field. Further, the decoherence of a system coupled to a spin environment with QPT has been investigated [12–14].

The quantum phase transition occurs at a level crossing point (gapless phase transition) or converged point of avoided level crossing point (gapped critical point) [23]. Because of the convergence of the energy levels at the critical point (CP), some special dynamic features may appear in the dynamic evolution of the central system in contact with an environment with QPT. It is shown that the disentanglement of the central system is greatly enhanced by the gapless quantum criticality of the environment [12]. Furthermore, the decoherence induced by the critical environment may display some universal features [13]. Since there exist separate states showing non-classical behaviour without entanglement [24–26], the quantum entanglement

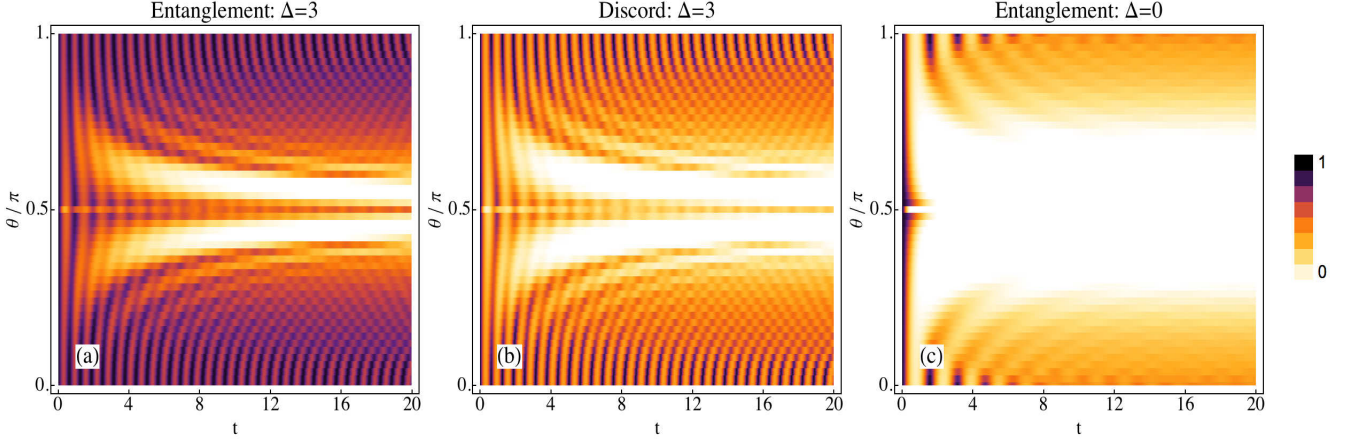


Fig. 1: (Color online) (a) Density plot of entanglement as a function of time, for $J_o = 1$, and $J_e = 4$. (b) The variation of the quantum discord as function of time t where the Hamiltonian parameters set as $J_o = 1$, and $J_e = 4$. (c) Entanglement evaluation for the gapless critical environment ($J_o = J_e = 1$). We consider $N = 400$ and weak-coupling between qubits and spin bath $g = 0.1$.

does not include all type of quantum correlations and their non-classical properties. Therefore, other measures of quantum correlations are expected.

A new but promising notion is quantum discord first introduced by Ollivier and Zurek [27], which can effectively capture all QCs in a quantum system. Recently, the dynamics of quantum discord under the effect of the environmental spin chain has increasingly been investigated [28]. The results show that quantum discord is more robust than entanglement for the system subjected to the environment [29].

In this letter, we consider two spins coupled to the one-dimensional general quantum compass model (GQCM) [30–33] in the presence of a transverse field [34–38]. GQCM is a simplified model that describes the nature of the orbital states in the case of a twofold degeneracy [34]. We analyze the effect of the gapped critical environment on the dynamic evolution of the two-spin entanglement, quantum discord, and negativity. Our comprehensive results show that induction of the gapped critical environment is nontrivially different than the gapless critical environment [13, 21], and strongly recommend it as a better entangler and quantum channel. In particular, in the weak coupling regime, QCs at the gapped CP decay rapidly from a maximum and after a very fast initial transient, start oscillating around an average value whereas show maximum decay away from CP. The results show that at CP an average value of QCs is enhanced with the increasing of the gap.

The full Hamiltonian which considers two spins coupled transversely to the spin 1/2 general quantum compass

chain is fully characterized by $\mathcal{H} = \mathcal{H}_E + \mathcal{H}_I$ with [12, 39]

$$\mathcal{H}_E = \sum_{i=1}^{N'} \left[J_o \tilde{\sigma}_{2i-1}^{(+)} \tilde{\sigma}_{2i}^{(+)} + J_e \tilde{\sigma}_{2i}^{(-)} \tilde{\sigma}_{2i+1}^{(-)} + h(\sigma_{2i-1}^z + \sigma_{2i}^z) \right],$$

$$\mathcal{H}_I = \frac{g}{2} (\sigma_A^z + \sigma_B^z) \sum_{i=1}^{N'} (\sigma_{2i-1}^z + \sigma_{2i}^z), \quad (1)$$

with \mathcal{H}_I describing the interaction between the central two qubits (σ_A^z, σ_B^z) and the surrounding spin chain with coupling strength g . Additionally, \mathcal{H}_E is the Hamiltonian of the environment describing the one-dimensional GQCM (1d-GQCM). In this representation, 1d-GQCM is constructed by antiferromagnetic order of X and Y pseudo-spin components on odd and even bonds at which the pseudo-spin operators are constructed as linear combinations of the Pauli matrices ($\sigma^{\alpha=x,y,z}$): $\tilde{\sigma}_{2i}^{(\pm)} = \tilde{\sigma}_i(\pm\theta) = \cos\theta\sigma_i^x \pm \sin\theta\sigma_i^y$ [38]. Here θ ($-\theta$) is arbitrary angle relative to σ^x for even (odd) bounds. J_e and J_o characterise the even and odd bound couplings respectively, h is a transverse field, and $N = 2N'$ is the number of spins.

We should emphasise that the 1d-GQCM is exactly solvable with the Jordan-Wigner transformation [34, 38, 40], which in momentum space leads to

$$\mathcal{H}_E = \sum_k \left[E_k^q (\gamma_k^{q\dagger} \gamma_k^q - \frac{1}{2}) + E_k^p (\gamma_k^{p\dagger} \gamma_k^p - \frac{1}{2}) \right], \quad (2)$$

where $\gamma_k^{p,q\dagger} (\gamma_k^{p,q})$ denote independent fermions creation (annihilation) operators. For states with even fermions $E_k^q = \sqrt{a + \sqrt{b}}$ and $E_k^p = \sqrt{a - \sqrt{b}}$, with $a = 4h^2 + |J|^2 + |L|^2$ and $b = (16h^2 + 2|J|^2)|L|^2 + J^2 L^{*2} + J^{*2} L^2$, where the parameters L and J are defined by $L = (J_o + J_e e^{i\theta})/4$, and $J = (J_o e^{i\theta} - J_e e^{i(k-\theta)})/4$.

It is known that the ground state is separated from the lowest-energy pseudo-spin excitation by a pseudo-spin gap

which vanishes at $\cos \theta_c = h/\sqrt{J_o J_e}$ [38]. First we concentrate on an idiosyncratic case of $\theta_c = \pi/2$ in the absence of external magnetic field. For this case, the ground-state has a macroscopic degeneracy of $2^{N/2-1}$ away from the isotropic point ($J_o \neq J_e$), which becomes $2^{N/2}$ when the orbital interactions are isotropic. The gap at $k = \pi$ is given by the anisotropy of the pseudo-spin exchange, $\Delta = |J_e - J_o|$, and just vanishes at $J_e = J_o$ which implies that the degeneracy increases by an additional factor of 2 due to the band-edge points. It has been proven that in the absence of an external magnetic field, for $\theta_c = \pi/2$, GQCM with Z_2 symmetry is critical for arbitrary J_e/J_o [38, 39]. QPT takes place between two different disordered phases where the model exhibits highest possible frustration of interactions [38, 39]. The isotropic point corresponds to a multicritical point where the gap closes quadratically at $k = \pm\pi$ as a result of the confluence of two Dirac points [41].

Since the central two qubits operators (σ_A^z, σ_B^z) and the environmental spin chain (σ_i^α) satisfy the commutation relation $[\sigma_A^z + \sigma_B^z, \sigma_i^\alpha] = 0$, the total Hamiltonian can be rewritten as $\mathcal{H} = \sum_{\mu=1}^4 |\varphi_\mu\rangle\langle\varphi_\mu| \otimes \mathcal{H}_E^{h_\mu}$. $|\varphi_\mu\rangle$, ($\mu = 1, \dots, 4$) denotes the μ th eigenstate of the operator $\frac{g}{2}(\sigma_A^z + \sigma_B^z)$ corresponding to the μ th eigenvalue ε_μ ($\varepsilon_1 = g, \varepsilon_2 = \varepsilon_3 = 0, \varepsilon_4 = -g$). Here $\mathcal{H}_E^{h_\mu}$ is defined through \mathcal{H}_E by replacing h with h_μ , where $h_\mu = h + \varepsilon_\mu$. We suppose that the initial state of the total system is disentangled with $\rho_{tot}(0) = \rho_{AB}(0) \otimes \rho_E(0)$. Considering $|\psi_E\rangle$ as an initial state of the environmental spin chains, and $\rho_{AB}(0)$ and $\rho_E(0) = |\psi_E\rangle\langle\psi_E|$ as initial density matrix state of the two-qubits system and environment respectively, the evolution of the total system will be governed by $\rho_{tot}(t) = U(t)\rho_{tot}(0)U^\dagger(t)$. Accordingly, the reduced density matrix of two-qubits AB is obtained by tracing out the environment [12],

$$\rho_{AB}(t) = \sum_{\mu, \nu=1}^4 F_{\nu\mu}(t) \langle\varphi_\nu|\rho_{AB}(0)|\varphi_\mu\rangle |\varphi_\nu\rangle\langle\varphi_\mu|, \quad (3)$$

where the decoherence factors are achieved by $F_{\nu\mu} = \langle\psi_E|U_E^{h_\mu\dagger}U_E^{h_\nu}|\psi_E\rangle$, and $U_E^{h_\nu} = U_E^{h_\nu}(t) = e^{-i\mathcal{H}_E^{h_\nu}t}$ is the time evolution operator driven by $\mathcal{H}_E^{h_\nu}$. By considering the ground state of the environment spin chain as an initial state, the decoherence factors reduce to a LE form given in Ref. [21], which is a dynamical version of the ground-state fidelity [40, 42].

We assume that the two qubits AB initially stem from the X -structure states $\rho_{AB}(0) = \frac{1}{4}(I_{AB} + \sum_\alpha c_\alpha \sigma_A^\alpha \otimes \sigma_B^\alpha)$ with the identity operator on two-qubits system, I_{AB} [43]. The parameters $c_{\alpha=x,y,z}$ are chosen to be real that insure that $\rho_{AB}(0)$ is a legal quantum state. This state is chosen in a general form to contain Bell-diagonal states and Werner states. According to Eq. (3), the reduced density matrix in the standard basis ($|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$) can be

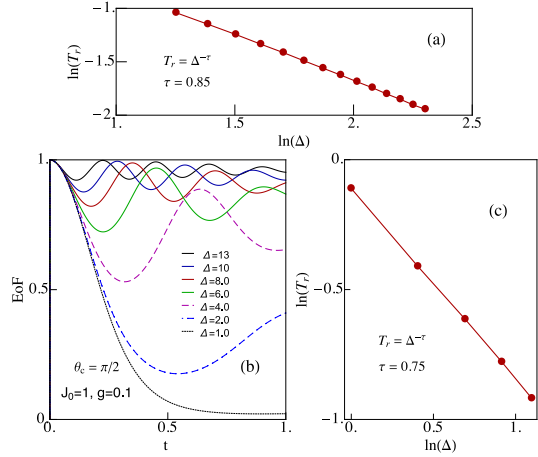


Fig. 2: (Color online) (a) The scaling behavior of the relaxation time T_r in terms of energy gap for $N = 400, J_o = 1, J_e = 4$ and $g = 0.1$. (b) Evaluation of entanglement as a function of time for different energy gap values. (c) The scaling behaviour of relaxation for small values of gap.

written as

$$\rho_{AB}(t) = \frac{1}{4} \begin{pmatrix} 1 + c_z & 0 & 0 & c_\beta \\ 0 & 1 - c_z & c_\gamma & 0 \\ 0 & c_\gamma^* & 1 - c_z & 0 \\ c_\beta^* & 0 & 0 & 1 + c_z \end{pmatrix}, \quad (4)$$

with $c_\beta = (c_x - c_y)F_{14}$, and $c_\gamma = (c_x + c_y)F_{23}$.

To quantify the entanglement dynamics of two qubits AB , we utilize the concurrence directly to calculate the entanglement of formation (EoF). The entanglement is a monotonically increasing function of concurrence (C'), which is defined by [44]

$$\text{EoF} = -f_{C'} \log_2 f_{C'} - (1 - f_{C'}) \log_2 (1 - f_{C'}),$$

with $f_{C'} = \frac{1}{2}(1 + \sqrt{1 - C'^2})$ and $C' = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$, where λ_i are the square roots of the eigenvalues in descending order of the operator $\rho_{AB}(t)(\sigma_A^y \otimes \sigma_B^y)\rho_{AB}^*(t)(\sigma_A^y \otimes \sigma_B^y)$. One can directly calculate the concurrence for the state defined by Eq. (4) as $C' = \max\{\frac{|c_\beta| + |c_z| - 1}{2}, \frac{|c_\gamma| - |c_z| - 1}{2}, 0\}$. Moreover, quantum discord is given by

$$Q_{AB} = \frac{1}{4} \sum_{\pm} \left[(1 - c_z \pm |c_\gamma|) \log_2 (1 - c_z \pm |c_\gamma|) + (1 + c_z \pm |c_\beta|) \log_2 (1 + c_z \pm |c_\beta|) \right] - C(\rho_{AB}(t)), \quad (5)$$

where $C(\rho_{AB}) = \sum_{\pm} \frac{1 \pm \vartheta}{2} \log_2 (1 \pm \vartheta)$, is the classical correlation with $\vartheta = \max\{|c_z|, \frac{|c_\beta| + |c_\gamma|}{2}\}$. To investigate the time evolution of QCs, we set the parameters $c_{x,z} = -c_y = 1$, so that the initial state becomes the Bell state $(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)/\sqrt{2}$. For this case, concurrence and quantum discord become $C' = |F_{14}(t)|$ and

$Q = \sum_{\pm} \frac{1 \pm |F_{14}(t)|}{2} \log_2(1 \pm |F_{14}(t)|)$, respectively. To illustrate the dynamical properties of QCs, we carry out the numerical calculation using the exact expression. The density plots of the time evolution of entanglement and quantum discord have been depicted in the weak coupling limit in Fig. 1(a & b). We set the Hamiltonian parameters in such a way that the system becomes gapped at CP $\theta_c = \pi/2$. Obviously both entanglement and quantum discord are monotonically increasing functions of the decoherence factor $|F_{14}(t)|$. Thus, quantum discord behaves in a similar way as entanglement does and is always less than entanglement in the process of evolution.

In contrast with the above situation, Fig. 1(c) represents a density plot of entanglement for the gapless critical case. As is clear, the enhanced decay of QCs induced by quantum criticality of the surrounding environment is broken by gapped quantum critical environment and the maximum decaying happens away from CP. The result shows that in the weak coupling regime, QCs decay from their maximum and after an initial transient, start oscillating around an average value. As observed in Fig. 1, QCs are symmetric with respect to $\theta = \pi/2$ and the numerical calculation shows that the valley narrows as g decreases and system size, N , increases. A more detailed analysis also shows that the relaxation time T_r at which QCs decay to their local minimums at CP of the environment (see Fig. 2(b)), reveals a power-law singularity as a function of the gap Δ . This is presented in Fig. 2(a) which specifies a linear behaviour of $\ln(T_r)$ versus $\ln(\Delta)$. The scaling behaviour is obtained as $T_r = |\Delta|^{-1/\tau}$ with the exponent $\tau = -0.75$ for small value of the gap whereas the exponent τ equals -0.85 for large values of the gap. Thus, one can conclude that the appearance of the two energy scales in the system is appropriate with the number of states which are involved. It means that the dynamics of QCs at very large energy gap values mainly originates from the ground state and the excited states have a very tiny contribution. It would be worth mentioning that the interaction coupling g does not affect the exponent τ in the weak coupling limit, as displayed by numerical simulations.

To study the scaling behaviour of QCs at CP, we have derived the scaling behaviour of local minimum values of QCs versus the second power of the relaxation time. This has been plotted in Fig. 3(a), which shows the linear behaviour of $\ln[\text{EoF}(T_r)]$, the same as $\ln[Q(T_r)]$, versus T_r^2 . In other words, the local minimum values of QCs scale exponentially with the second power of the relaxation time, $\text{EoF}(T_r) = \exp(-\delta T_r^2)$ with exponent $\delta \propto (-\Delta/g)$. In particular, our results imply that the decay of the local minimum of QCs at CP of environment enhances with decreasing energy gap and increasing interaction coupling (see Fig. 3(b)).

However, oscillations of QCs around an average value increases as the energy gap of the environmental spin chain increases. On the other hand, Fig. 3(c) reveals an interesting phenomenon in QCs at CP ($\theta_c = \pi/2$).

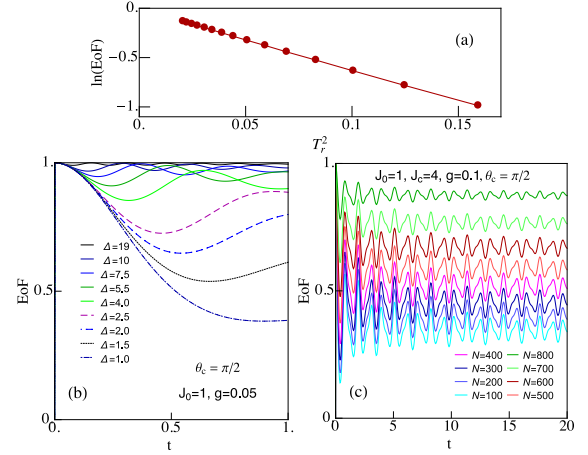


Fig. 3: (Color online) (a) Scaling of the minimum of quantum correlations at critical point ($\theta_c = \pi/2$) for systems of various gaps (Fig. 2b). (b) Entanglement evaluation as a function of time for different energy gap for $N = 400$, $J_0 = 1$, and $g = 0.05$. (c) Entanglement as a function of time at critical point ($\theta_c = \pi/2$) for various system sizes, and $J_0 = 1$, $J_e = 4$, and $g = 0.1$.

The periods of the revival of QCs is independent of the size of the environment which means that CP is a scale invariance point where quantum fluctuations extend over all length scales. Moreover, a similar result also has confirmed the above argument for the time evolution of QCs from mixed states $c_x = 1$, and $c_y = -c_z \in [0, 1]$. The presented results presented here indicate that in the strong coupling regime decoherence factors and QCs decay to zero in a very short time as opposed to what happens in the weak coupling regime. As the coupling strength increases, the valleys widen and the influence of the energy gap on the generation of QCs decreases even when approached along the gapped critical point.

In the presence of a magnetic field, GQCM shows a gapless phase transition and our results support previous studies [12–16]. We also examine two qutrits coupled to one 1d-GQCM [45]. The negativity displays almost the same dynamical behaviour as entanglement and quantum discord do.

In summary, using the general quantum compass model as an environmental system, the dynamical evolution of the decoherence factors, quantum correlations, and negativity of the central spins has been investigated for different initial states. The relation between the quantum-classical transition of the central system, and the occurrence of an avoided level crossing quantum phase transition in its surrounding system has been analysed. It is well known that the gapless quantum criticality enhances decaying of decoherence factors, while our calculations repre-

sent a different story for gapped critical environment. The finding results show that long-time quantum correlations at the critical point is an effect of gapped criticality, and maximum decaying occurs away from the critical point.

The role of the gapped critical spin chain is to prevent the complete drain of information from central systems to the environment [46] and provides them a better environment for preserving quantum correlations. In other words, the amount of decoherence which travels into the central spin state depends on the excited states of the environment. Hence the energy gap could block the propagation of decoherence along the environment and consequently reduces its effect on the central spin. These results highlight the current outlook of using quantum spin chains as entanglers or quantum channels in quantum information devises [47, 48]. Besides, quantum gapped criticality may have potential applications in quantum computations.

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REFERENCES

- [1] A. Barenco and A. K. Ekert, 1995 *Journal of Modern Optics*, 42(6) 1253–1259.
- [2] S. F. Pereira, Z. Y. Ou, and H. J. Kimble, 2000 *Phys. Rev. A*, 62 042311.
- [3] A. Barenco, 1996 *Contemporary Physics*, 37(5) 375–389.
- [4] L. K. Grover, 1997 *Phys. Rev. Lett.*, 79 325–328.
- [5] R. Jafari, 2010 *Phys. Rev. A*, 82 052317.
- [6] A. Einstein, B. Podolsky, and N. Rosen, 1935 *Phys. Rev.*, 47 777–780.
- [7] J. S. Bell, 1964 *Physics*, 1 195.
- [8] W. H. Zurek, 2003 *Rev. Mod. Phys.*, 75 715–775.
- [9] J. P. Paz. and W. H. Zurek, 2001 in *Coherent Matter Waves*, pages 533–614. EDP Sciences, Springer Verlag, Berlin.
- [10] M. P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. P. Walborn, P. H. S. Ribeiro, and L. Davidovich, 2007 *Science*, 316(5824) 579–582.
- [11] C. Cormick and J. Pablo Paz, 2008 *Phys. Rev. A*, 77 022317.
- [12] Z.-G. Yuan, P. Zhang, and S.-S. Li, 2007 *Phys. Rev. A*, 76 042118.
- [13] F. M. Cucchietti, S. Fernandez-Vidal, and J. P. Paz, 2007 *Phys. Rev. A*, 75 032337.
- [14] S. Mostame, G. Schaller, and R. Schützhold, 2007 *Phys. Rev. A*, 76 030304.
- [15] Z. Sun, X. Wang, and C. P. Sun, 2007 *Phys. Rev. A*, 75 062312.
- [16] W. W. Cheng and J.-M. Liu, 2009 *Phys. Rev. A*, 79 052320.
- [17] B.-Q. Liu, B. Shao, and J. Zou, 2009 *Phys. Rev. A*, 80 062322.
- [18] You, W. L. and Dong, Y. L., 2010 *Eur. Phys. J. D*, 57(3) 439–445.
- [19] M.-L. Hu, 2010 *Physics Letters A*, 374(34) 3520 – 3528.
- [20] D.-W. Luo, H.-Q. Lin, J.-B. Xu, and D.-X. Yao, 2011 *Phys. Rev. A*, 84 062112.
- [21] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, 2006 *Phys. Rev. Lett.*, 96 140604.
- [22] D. Rossini, T. Calarco, V. Giovannetti, S. Montangero, and R. Fazio, 2007 *Phys. Rev. A*, 75 032333.
- [23] S. Sachdev, 2011 *Quantum Phase Transitions*. Cambridge University Press, second edition. Cambridge Books Online.
- [24] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke, 2005 *Phys. Rev. A*, 71 062307.
- [25] J. Niset and N. J. Cerf, 2006 *Phys. Rev. A*, 74 052103.
- [26] A. Datta, S. T. Flammia, and C. M. Caves, 2005 *Phys. Rev. A*, 72 042316.
- [27] H. Ollivier and W. H. Zurek, 2001 *Phys. Rev. Lett.*, 88 017901.
- [28] B.-Q. Liu, B. Shao, and J. Zou, 2010 *Phys. Rev. A*, 82 062119.
- [29] Guo, Jin-Liang and Long, Gui-Lu, 2013 *Eur. Phys. J. D*, 67(3) 53.
- [30] E. Eriksson and H. Johannesson, 2009 *Phys. Rev. B*, 79 224424.
- [31] Z. Nussinov and G. Ortiz, 2009 *Phys. Rev. B*, 79 214440.
- [32] Z. Nussinov and J. van den Brink, 2013 *arXiv:1303.5922*.
- [33] Mahdavi-far, S., 2010 *Eur. Phys. J. B*, 77(1) 77–82.
- [34] R. Jafari, 2011 *Phys. Rev. B*, 84 035112.
- [35] Jafari, R., 2012 *Eur. Phys. J. B*, 85(5) 167.
- [36] W.-L. You, G.-H. Liu, P. Horsch, and A. M. Oleś, 2014 *Phys. Rev. B*, 90 094413.
- [37] Motamedifar, M., Mahdavi-far, S., and Farjami Shayesteh, S., 2011 *Eur. Phys. J. B*, 83(2) 181–189.
- [38] W.-L. You, P. Horsch, and A. M. Oleś, 2014 *Phys. Rev. B*, 89 104425.
- [39] Z. Nussinov and J. van den Brink, 2015 *Rev. Mod. Phys.*, 87 1–59.
- [40] R. Jafari, 2015 *arXiv:1503.08719*.
- [41] Y. Niu, S. B. Chung, C.-H. Hsu, I. Mandal, S. Raghu, and S. Chakravarty, 2012 *Phys. Rev. B*, 85 035110.
- [42] S. Montes and A. Hamma, 2012 *Phys. Rev. E*, 86 021101.
- [43] B. G. Englert and N. Metwally, 2001 *Rev. Mod. Phys.*, 72(1) 35–42.
- [44] W. K. Wootters, 1998 *Phys. Rev. Lett.*, 80 2245–2248.
- [45] Z. Sun, X. Wang, and C. P. Sun, 2007 *Phys. Rev. A*, 75 062312.
- [46] G. H. Aguilar, O. J. Farias, A. Valdes-Hernandez, P. H. Souto Ribeiro, L. Davidovich, and S. P. Walborn, 2014 *Phys. Rev. A*, 89 022339.
- [47] L. d. Rio, J. Aberg, R. Renner, O. Dahlsten, and V. Vedral, 2011 *Nature*, 474(7349) 61–63.
- [48] S. Sahling, G. Remenyi, C. Paulsen, P. Monceau, V. Saligrama, C. Marin, A. Revcolevschi, L. P. Regnault, S. Raymond, and J. E. Lorenzo, 2015 *Nat Phys*, 11(3) 255–260.